ing to the interaction (12), will, therefore, be predominantly spin-independent for large separations and low energies. In the approximation in which baryon mass differences are neglected,[35] the Λ-nucleon TPEP corresponding to the interaction (12) is just the TPEP obtained by Gupta[17] with $\boldsymbol{\tau}(1)\cdot\boldsymbol{\tau}(2)=0$ and $g_N^4=g_N^2g_{\Lambda\Sigma}^2$.[3,4,6] In the region of small separations, the form of the Λ-nucleon potential is presumably determined by meson-exchange mechanisms not considered here. The work reported here indicates that the spin dependence of the two-body Λ-nucleon interaction at low energies is primarily an attribute of the interior region of the potential, whose parameters will probably have to be determined phenomenologically.[36]

[35] Our calculations indicate that the neglect of mass differences may not be a very good approximation. For example, the static values of the integrals I_i (see footnote 26) are all of comparable magnitude, and they contribute to the M matrix coefficients in an additive manner; and yet I_3, I_4, I_5, and I_6 vanish when mass differences are neglected. Moreover, about one-third of the static value of I_2, which determines the M matrix for zero-energy scattering, arises from the terms in the integrand (20c) which are proportional to the $\Sigma-\Lambda$ mass difference $(\alpha-1)K_\Lambda$.

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New Series for Phase Shift in Potential Scattering

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A new series for the phase shift has been derived for the Schrödinger, Klein-Gordon, and Dirac equations. This series converges faster than the Born series for the tangent of the phase shift. This is so because the sum of the first n terms in the new series includes exactly all the terms up to the $2(2^n-1)$th order in the Born series. Under the condition which is tantamount to that the phase shift cannot be larger than 63°, the series converges absolutely. At high energies the series can be analytically continued with respect to the strength of the potential beyond such a limit. It is shown that the high-energy limit of the phase shift is given by its first Born approximation and that the difference between even and noneven potentials is reflected in the respective phase shifts to all orders.

1. INTRODUCTION

THE high-energy potential scattering has been studied extensively[1–7] using the Schrödinger, the Klein-Gordon, or the Dirac equation. Recently, the interest in high-energy potential scattering has been revived[8–10] with the hope that it may be possible to suggest something useful to the high-energy field theoretical scattering. It is the purpose of this paper to derive a new series for the phase shift and apply the new formula to high-energy scattering. This series gives a unique value for the phase shift based on its Born expansion. A result which carries important theoretical implications is obtained, but it is not our primary concern to improve on a practical method of computing a phase shift from a given potential.

It has been customary to start with a formula for the tangent of the phase shift or something equivalent to it. In this case the phase shift is determined only up to an arbitrary multiple of π. Therefore, two sets of phase shifts, which differ by an arbitrary step function of momentum whose value takes only some multiple of π, are equivalent to each other. Such an ambiguity cannot be removed when one uses the tangent of the phase shift.

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It is natural, however, to put the phase shift null when there is no potential. As the strength of the potential is increased adiabatically, a unique value for the phase shift will be obtained by summing the Born series for it. This is what can be termed the absolute definition of the phase shift.[11]

When one is concerned with the Schrödinger equation, the high-energy limit of the phase shift vanishes. For both the Dirac and the Klein-Gordon equations, the high-energy limit for a finite angular momentum is

$$\delta(\infty)=-\int_0^\infty V(r)dr, \tag{1.1}$$

provided the above integral does exist. The limit (1.1) has been derived by Parzen[2] for the Dirac equation; later he derived a relation between the Dirac phase shift and the Klein-Gordon phase shift,[12] from which the same high-energy limit (1.1) follows for the Klein-Gordon phase shift. If the potential falls off fast enough at large distances, and if there is no bound state, the Born series for the phase shift can be safely used at zero energy. From this it follows that the phase shift vanishes in the limit of zero energy for all equations. With its value at both limits fixed in this way, the phase shift gives a measure of the strength of the interaction; a larger phase shift follows from a stronger interaction. The feasibility of the absolute definition of the phase shift has been pointed out in connection with the derivation[13] of the straightforward relation between the phase shift and the generating function (the exponent) of the transformation function. The latter transforms a free state into a corresponding interacting state.

If the potential supports N_l bound states of angular momentum l, then according to Levinson's theorem[14] the zero-energy phase shift for the Schrödinger equation is equal to $N_l\pi$. Such a jump of the zero-energy phase shift can be accounted for by the orthogonality of scattering states to a bound state.[15] A bound state does not affect the phase shift at high energies[15] with which we are mainly interested in this paper; therefore, we assume the absence of a bound state. However, it is worthwhile emphasizing, that the observation of the over-all behavior of the phase shift may lead to a deeper insight into the structure of the available Hilbert space.[16]

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An advantage of working with the series for the phase shift itself is that we can easily define the phase shift with magnitude larger than $\frac{1}{2}\pi$, provided the series can be summed. This is to be compared to the Born series for the tangent of the phase shift which will be outside its radius of convergence under such a circumstance. The right side of (1.1) may take any value depending on the strength of the potential. Therefore, if the magnitude of the right side of (1.1) is larger than $\frac{1}{2}\pi$, we can derive the high-energy limit of the phase shift easily from our new series, without going through the analytical continuation with respect to the strength of the potential. This would be necessary if we had started with the tangent of the phase shift. Thus, the new series may serve as a starting point for the investigation of the over-all behavior of the phase shift as a function of energy.

It will be seen in Sec. 4 that (1.1) follows immediately from the fact that the first Born approximation for the phase shift is very good at high energies, whatever the strength of the potential. The new series is somewhat more advanced than a simple Born series. A compact form is derived for a partial sum of the original Born series. Actually, the sum of n terms in the new series includes exactly all the terms up to the $2(2^n-1)$th order in the Born series. Therefore, the advantage of the new series is more pronounced as n becomes larger.

The class of potentials, termed noneven potentials, for which a derivative of odd order does not vanish at the origin has been studied in reference 7. The asymptotic form of the high-energy amplitude off the forward direction is an inverse power series in the momentum transfer. In contrast to this, the asymptotic amplitude for even potentials (for which all derivatives of odd order vanish at the origin) decreases faster than any inverse power of the momentum transfer. The behavior of Regge poles at high energies shows a remarkable difference between even and noneven potentials as has been shown by Bethe and Kinoshita.[9] This distinction is very important for the physical understanding of high-energy elementary particle scattering in terms of an optical model, for instance. A theorem has been demonstrated in Sec. 4, which reflects the even-noneven difference in their respective phase shifts. As the noneven potentials are now reasonably well understood, it would be desirable to extend the analysis to the even potentials. We do not attempt to extend the analysis to the even potentials. We do not attempt at any extensive application of the new series in this paper, but it will be a helpful tool in the derivation of an exact asymptotic expansion of the higher order phase shifts at high energy.

In Sec. 2 a new series for the phase shift will be derived. Some of its properties will be discussed in Sec. 3. In Sec. 4, besides the result concerning the even-noneven difference, the high-energy asymptotic expansion of the second- and the third-order phase shift has been presented for the S wave.

2. NEW SERIES FOR PHASE SHIFT

The radial Schrödinger equation for a spherically symmetric potential $V(r)$ is given by

$$\frac{d^2\psi}{dr^2}+\left[p^2-\frac{l(l+1)}{r^2}-U(r)\right]\psi=0, \tag{2.1}$$

where $U=2mV$, m is the mass of the particle, and $\psi(r)$ is r times the radial wave function. Equation (2.1) can be cast into the linearized matrix form by setting

$$\Phi=\begin{pmatrix}\psi\\ d\psi/d(pr)\end{pmatrix} \tag{2.2}$$

and

$$d\Phi/dr=H\Phi, \tag{2.3}$$

where the matrix H is given by

$$H=\frac{1}{p}\begin{bmatrix}0 & p^2\\ \dfrac{l(l+1)}{r^2}+U(r)-p^2 & 0\end{bmatrix}. \tag{2.4}$$

[Equation (2.3) has the same form as the time-dependent Schrödinger equation.] In order to eliminate the centrifugal potential we shall split the matrix H and set

$$H=H_c+H_p, \tag{2.5}$$

where

$$H_c=\frac{1}{p}\begin{bmatrix}0 & p^2\\ \dfrac{l(l+1)}{r^2}-p^2 & 0\end{bmatrix} \tag{2.6}$$

and

$$H_p=\frac{1}{p}\begin{pmatrix}0 & 0\\ U & 0\end{pmatrix}. \tag{2.7}$$

If we then put $\Phi=W\chi$, we obtain from (2.3)

$$W\frac{d\chi}{dr}+\frac{dW}{dr}\chi=(H_c+H_p)W\chi. \tag{2.8}$$

Now if the transformation function W satisfies the equation

$$dW/dr=H_cW, \tag{2.9}$$

then the vector χ must be a solution of

$$d\chi/dr=(W^{-1}H_pW)\chi. \tag{2.10}$$

It is easy to verify that (2.9) gives

$$W=\begin{pmatrix}u_l(pr) & v_l(pr)\\ u_l'(pr) & v_l'(pr)\end{pmatrix}, \tag{2.11}$$

where u_l and v_l, the regular and irregular solutions of the potential free-wave equation, respectively, are the Riccati-Bessel functions. [We have put a multiplicative constant in (2.11) equal to one because it may always be absorbed by χ.] Therefore, Eq. (2.10) reads

$$\begin{aligned}\frac{d\chi}{dr}&=\begin{pmatrix}-M_{uv}{}^{(0)} & -M_{vv}{}^{(0)}\\ M_{uu}{}^{(0)} & M_{uv}{}^{(0)}\end{pmatrix}\cdot\chi\\ &=\Big[\tfrac{1}{2}(M_{uu}{}^{(0)}-M_{vv}{}^{(0)})\sigma_1\\ &\qquad-\frac{i}{2}(M_{uu}{}^{(0)}+M_{vv}{}^{(0)})\sigma_2-M_{uv}{}^{(0)}\sigma_3\Big]\cdot\chi,\end{aligned} \tag{2.12}$$

where we have used the following abbreviations:

$$\begin{aligned}M_{uu}{}^{(0)}(r)&=\frac{1}{p}u_l(pr)U(r)u_l(pr),\\ M_{vv}{}^{(0)}(r)&=\frac{1}{p}v_l(pr)U(r)v_l(pr),\\ M_{uv}{}^{(0)}(r)&=\frac{1}{p}u_l(pr)U(r)v_l(pr),\end{aligned} \tag{2.13}$$

and σ_i are the three Pauli matrices.

We shall now perform two successive transformations on χ.

$$\chi=\Omega_a{}^{(1)}\Omega_p{}^{(1)}\chi_1, \tag{2.14}$$

such that the transformed vector χ_1 will satisfy an equation which is similar in form to (2.12) but has coefficients M replaced by quantities of higher order in U. The transformations Ω_a and Ω_p induce modulations in the amplitude and the phase of the wave function ψ, respectively. We can eliminate the third component in (2.12) by using $\Omega_a{}^{(1)}$ which satisfies the equation

$$d\Omega_a{}^{(1)}/dr=-M_{uv}{}^{(0)}\sigma_3\Omega_a{}^{(1)}. \tag{2.15}$$

Its solution is

$$\begin{aligned}\Omega_a{}^{(1)}&=\exp[\sigma_3\lambda^{(0)}(r)]\\ &=\cosh[\lambda^{(0)}(r)]+\sigma_3\sinh[\lambda^{(0)}(r)],\end{aligned} \tag{2.16}$$

where

$$\lambda^{(0)}(r)=\int_r^\infty M_{uv}{}^{(0)}(r')dr'. \tag{2.17}$$

Thus, if we call $\chi_1'=\Omega_p{}^{(1)}\chi_1$, then χ_1' has to satisfy the equation

$$d\chi_1'/dr=[-\tfrac{1}{2}U^{(1)}(r)(\sigma_1+i\sigma_2)-i\sigma_2K^{(1)}(r)]\chi_1', \tag{2.18}$$

where we have used the following notations

$$\begin{aligned}K^{(1)}(r)&=M_{uu}{}^{(0)}(r)\exp[2\lambda^{(0)}(r)],\\ L^{(1)}(r)&=M_{vv}{}^{(0)}(r)\exp[-2\lambda^{(0)}(r)],\\ U^{(1)}(r)&=L^{(1)}(r)-K^{(1)}(r).\end{aligned} \tag{2.19}$$

Now if $\Omega_p^{(1)}$ solves the equation

$$d\Omega_p^{(1)}/dr = -i\sigma_2 K^{(1)}(r)\Omega_p^{(1)}, \tag{2.20}$$

then the second term on the right side of (2.18) will be eliminated and the transformed vector χ_1 has to satisfy

$$d\chi_1/dr = -\tfrac{1}{2}U^{(1)}(r)[(\Omega_p^{(1)})^{-1}(\sigma_1+i\sigma_2)\Omega_p^{(1)}]\cdot\chi_1. \tag{2.21}$$

Equation (2.20) gives for $\Omega_p^{(1)}$

$$\begin{aligned}\Omega_p^{(1)} &= \exp[-i\sigma_2\xi^{(1)}(r)],\\ &= \cos\xi^{(1)}(r) - i\sigma_2 \sin\xi^{(1)}(r),\end{aligned} \tag{2.22}$$

where

$$\xi^{(1)}(r) = \int_0^r K^{(1)}(r')dr'. \tag{2.23}$$

Hence, (2.21) takes the form

$$\frac{d\chi_1}{dr} = \Big[\tfrac{1}{2}(M_{uu}^{(1)} - M_{vv}^{(1)})\sigma_1 - \frac{i}{2}(M_{uu}^{(1)} + M_{vv}^{(1)})\sigma_2 - M_{uv}^{(1)}\sigma_3\Big]\chi_1, \tag{2.24}$$

where

$$\begin{aligned} M_{uu}^{(1)} &= \sin\xi^{(1)}\ U^{(1)} \sin\xi^{(1)},\\ M_{vv}^{(1)} &= \cos\xi^{(1)}\ U^{(1)} \cos\xi^{(1)},\\ M_{uv}^{(1)} &= \sin\xi^{(1)}\ U^{(1)} \cos\xi^{(1)}.\end{aligned} \tag{2.25}$$

This equation is the same in form as the original equation for χ, (2.12). The new equation is obtained from (2.12) by the substitution $M^{(0)} \to M^{(1)}$, or more explicitly

$$\begin{aligned} u_l &\to \sin\xi^{(1)},\\ v_l &\to \cos\xi^{(1)},\\ U &\to U^{(1)}.\end{aligned} \tag{2.26}$$

We call the transformation which leads from (2.12) to (2.24) the first integration cycle. The similarity between (2.12) and (2.24) is very important as it enables one to write down immediately the higher order transformations. In complete analogy to the first integration cycle, the functions $\lambda^{(1)}(r)$, $\xi^{(2)}(r)$, and $M^{(2)}$ which determine the second integration cycle are given by

$$\begin{aligned}\lambda^{(1)}(r) &= \int_r^\infty M_{uv}^{(1)}dr',\\ \xi^{(2)}(r) &= \int_0^r K^{(2)}dr',\end{aligned} \tag{2.27}$$

where

$$K^{(2)}(r) = M_{uu}^{(1)}\exp[2\lambda^{(1)}]. \tag{2.28}$$

The functions $M^{(2)}$ are given by (2.25) in which $\xi^{(1)}$ is replaced by $\xi^{(2)}$ and $U^{(1)}$ is replaced by

$$U^{(2)} = L^{(2)} - K^{(2)},$$

with

$$L^{(2)} = M_{vv}^{(1)}\exp[-2\lambda^{(1)}]. \tag{2.29}$$

The two transformation functions, $\Omega_a^{(2)}$ and $\Omega_p^{(2)}$ are given by (2.16) and (2.22) in which $\lambda^{(0)}$ and $\xi^{(1)}$ are replaced by $\lambda^{(1)}$ and $\xi^{(2)}$, respectively. It is now an easy matter to write down the functions $\lambda^{(n-1)}$ and $\xi^{(n)}$ of the nth cycle. They are determined by the following recursion formulas, valid for $n>1$

$$\lambda^{(n-1)} = \frac{1}{2}\int_r^\infty \sin 2\xi^{(n-1)}(r')U^{(1)}(r')\Lambda_{n-2}(r')dr', \tag{2.30}$$

$$\xi^{(n)} = \int_0^r \sin^2\xi^{(n-1)}(r')U_{n-1}(r')\Lambda_{n-2}(r')dr', \tag{2.31}$$

where

$$U_n = U^{(1)}\exp[2\lambda^{(n)}]. \tag{2.32}$$

$$\frac{\Lambda_n}{\Lambda_{n-1}} = \cos^2\xi^{(n)}\exp[-2\lambda^{(n)}] - \sin^2\xi^{(n)}\exp[2\lambda^{(n)}] \tag{2.33}$$

with $\Lambda_0 = 1$. These formulas enable one to calculate step by step the functions $\lambda^{(n)}$ and $\xi^{(n)}$ to any desired order. The transformation functions, $\Omega_a^{(n)}$ and $\Omega_p^{(n)}$ of the nth integration cycle are then given by

$$\Omega_a^{(n)} = \cosh[\lambda^{(n-1)}] + \sigma_3 \sinh[\lambda^{(n-1)}] \tag{2.34}$$

$$\Omega_p^{(n)} = \cos[\xi^{(n)}] - i\sigma_2 \sin[\xi^{(n)}]. \tag{2.35}$$

We now come to the discussion of the boundary conditions for the functions $\lambda^{(n)}$ and $\xi^{(n)}$ and the determination of the phase shift. Let us concentrate on the first integration cycle. First, we note that if we call

$$\chi_1 = \begin{pmatrix} C_1(r) \\ -S_1(r)\end{pmatrix}, \tag{2.36}$$

then the wave function after the first cycle is given by

$$\begin{aligned}\psi(r) = u_l(pr)[&C_1(r)\cos\xi^{(1)}(r) + S_1(r)\sin\xi^{(1)}(r)]\exp[2\lambda^{(0)}(r)]\\ + v_l(pr)[&C_1(r)\sin\xi^{(1)}(r) - S_1(r)\cos\xi^{(1)}(r)]\exp[-2\lambda^{(0)}(r)].\end{aligned} \tag{2.37}$$

Now in order to make this solution regular everywhere, the coefficient of v_l has to vanish at $r=0$. In (2.22) we have chosen $\xi^{(1)}(0)=0$. Therefore, we require $S_1(0)=0$. From the differential equation for χ_1, (2.24), it then follows in conjunction with the mentioned boundary condition for S_1 that C_1 starts with a term of zeroth order in the potential, whereas S_1 starts with a term of third order at least. This is easily proved if one remembers that $M_{vv}^{(1)}$, $M_{uv}^{(1)}$, and $M_{uu}^{(1)}$ are of first, second, and third order in the potential, respectively. Therefore, if one wants to consider the wave function up to the second order in the potential only, one may put identically

$$S_1(r) = 0. \tag{2.38}$$

Thus, the wave function ψ is proportional to

$$u_l(pr)\cos\xi^{(1)}(r)\exp[2\lambda^{(0)}(r)] + v_l(pr)\sin\xi^{(1)}(r)\exp[-2\lambda^{(0)}(r)] \quad (2.39)$$

which becomes asymptotically equal to

$$\sinh 2\lambda^{(0)}(r)\times\sin[pr-\tfrac{1}{2}\pi l+\xi^{(1)}(r)] + \cosh 2\lambda^{(0)}(r)\times\sin[pr-\tfrac{1}{2}\pi l-\xi^{(1)}(r)], \quad (2.40)$$

since the asymptotic forms of the Riccati-Bessel functions for large r are

$$u_l(pr)\sim\sin(pr-\tfrac{1}{2}\pi l)$$
$$v_l(pr)\sim-\cos(pr-\tfrac{1}{2}\pi l). \quad (2.41)$$

The phase shift δ_l is now determined by the requirement that the wave function behaves for large r like

$$\psi(pr)\sim\sin[pr-\tfrac{1}{2}\pi l+\delta_l]. \quad (2.42)$$

It follows that in order to compel the wave function (2.40) to behave correctly for large r, one has to impose the boundary condition

$$\lambda^{(0)}(\infty)=0. \quad (2.43)$$

This has been tacitly assumed in its derivation, (2.17), Comparison of (2.40) with (2.42) then gives for the phase shift up to the second order in the potential

$$\delta_l=-\eta_l^{(1)},$$

where

$$\eta_l^{(1)}=\xi^{(1)}(\infty)=\frac{1}{p}\int_0^\infty u_l^2(pr)U(r) \times\exp\left[\frac{2}{p}\int_r^\infty u_l(pr')v_l(pr')U(r')dr'\right]dr. \quad (2.44)$$

The last step follows from (2.23), (2.19), (2.13), and (2.17). Putting the exponential factor equal to one, the first-order Born approximation for the phase shift is obtained. The next term in the expansion reproduces the second-order Born approximation. The higher order terms are only partly reproduced. This of course was to be expected because the wave function, (2.39), is correct only up to the second order in the potential. The first term (2.44) has been derived also by Calogero.[17] The higher order terms of the new expansion are derived in a completely analogous way. For example, knowing that $\xi^{(2)}(0)=0$, it is easy to derive from the differential equation for

$$\chi_2(r)=\begin{pmatrix} C_2(r) \\ -S_2(r)\end{pmatrix}$$

[which is the same as (2.24) except that $M^{(1)}$ is replaced by $M^{(2)}$] that terms of the lowest order for C_2 and S_2 are of zeroth and seventh order in the potential, respectively. Hence, if one considers the phase shift up to the sixth order only one may put $S_2(r)=0$ identically. Then in order to give the wave function for large r the form (2.42) one has to require $\lambda^{(1)}(\infty)=0$. Therefore, the solution Φ of (2.3) becomes for large r

$$\Phi=\lim_{r=\infty} W\Omega_p^{(1)}\cdot\Omega_p^{(2)}\begin{pmatrix}1\\0\end{pmatrix},$$

and the wave function is given by

$$\psi\underset{r=\infty}{\longrightarrow} u_l\cos[\xi^{(1)}+\xi^{(2)}]+v_l\sin[\xi^{(1)}+\xi^{(2)}].$$

Therefore, the phase shift up to the sixth order becomes

$$\delta_l=-[\eta_l^{(1)}+\eta_l^{(2)}], \quad (2.45)$$

where $\eta^{(1)}$ is given by (2.44) and

$$\eta_l^{(2)}=\xi^{(2)}(\infty)=\int_0^\infty \sin^2\xi^{(1)}(r)U^{(1)}(r) \times\exp\left[\int_r^\infty \sin 2\xi^{(1)}(r')U^{(1)}(r')dr'\right]dr. \quad (2.46)$$

The last step follows from (2.27), (2.28), and (2.25). The functions $\xi^{(1)}(r)$ and $U^{(1)}(r)$ are given by (2.23) and (2.19), respectively. By the same procedure it is a simple matter to show that for every n we have

$$\lambda^{(n)}(\infty)=0, \quad (2.47)$$

Hence, to an arbitrary order of the potential the solution Φ for large values of r is given by the infinite product of transformation functions

$$\Phi=\lim_{r=\infty} W\prod_{n=1}^{\infty}\Omega_p^{(n)}\begin{pmatrix}1\\0\end{pmatrix} \quad (2.48)$$

which for the wave function leads to

$$\psi\longrightarrow u_l\cos\sum_{n=1}^{\infty}\xi^{(n)}(\infty)+v_l\sin\sum_{n=1}^{\infty}\xi^{(n)}(\infty).$$

Therefore, we obtain finally for the phase shift the infinite series

$$\delta_l=-\sum_1^\infty \eta^{(n)},$$

where

$$\eta^{(n)}=\xi^{(n)}(\infty) \quad (2.49)$$

and $\xi^{(n)}$ for $n>1$ is determined by the recursion formula (2.31), and $\eta^{(1)}$ is given by (2.44). Some properties of the new series will be discussed in the next two sections.

Some remarks are due here as to the physical interpretation of the transformation functions Ω_a and Ω_p. Consider the solution Φ which takes the asymptotic form (2.48) for large r. In view of the general form of Ω's, (2.34), and (2.35), as well as the general form of ξ,

[17] F. Calogero, Nuovo Cimento **27**, 261 (1963).

(2.31), this solution Φ assumes at the origin the form

$$\Phi=\lim_{r=0} W \prod_{n=1}^{\infty} \Omega_a{}^{(n)}\begin{pmatrix}1\\0\end{pmatrix}=\lim_{r=0} W\begin{pmatrix}\exp \sum_{n=0}^{\infty} \lambda^{(n)}(0)\\ 0\end{pmatrix}. \quad (2.50)$$

On the other hand, without the potential the solution with the same asymptotic amplitude for large r is given simply by

$$\Phi_f=\lim_{r=0} W\begin{pmatrix}1\\0\end{pmatrix} \quad (2.51)$$

at the origin. Therefore, the amplitude at the origin is reduced by the ratio $\exp[\sum_{n=0}^{\infty} \lambda^{(n)}(0)]:1$ relatively to the case without potential. This ratio is the inverse of the amplitude of the Jost function.[18] Therefore, we obtain

$$-\ln|f_l(p)|=\sum_0^{\infty} \lambda^{(n)}(0), \quad (2.52)$$

where $f_l(p)$ is the Jost function. Thus, a partially summed Born series has been obtained also for the logarithm of the amplitude of the Jost function. We, therefore, conclude that the operation of Ω_a modulates the amplitude of the wave function ψ at the origin. On the other hand, the transformation function Ω_p causes only the phase modulation of the wave function.

The foregoing method for the derivation of the phase shift of the Schrödinger equation applies to the Klein-Gordon equation if the definition of U is modified. It may equally well be applied to the Dirac equation. In this case the radial wave function is of the following form[19]:

$$\frac{d}{dr}\begin{pmatrix}f_l\\g_l\end{pmatrix}=\begin{bmatrix} -\dfrac{l+1}{r} & [E-m-V(r)] \\ -[E+m-V(r)] & \dfrac{l+1}{r}\end{bmatrix}\begin{pmatrix}f_l\\g_l\end{pmatrix}. \quad (2.53)$$

It is easy to see that the matrix, W_D, which eliminates the centrifugal potential and, hence, is a solution of (2.53) for $V(r)=0$, is given by

$$W_D=\begin{pmatrix}u_{l+1} & v_{l+1}\\ u_l & v_l\end{pmatrix}. \quad (2.54)$$

Therefore, the equation for χ_D, which replaces (2.10) and (2.12) assumes the form

$$\frac{d\chi_D}{dr}=iV(r)(W_D{}^{-1}\sigma_2 W_D)\chi_D$$
$$=V(r)\begin{pmatrix}-(u_lv_l+u_{l+1}v_{l+1}) & -(v_l^2+v_{l+1}^2)\\ (u_l^2+u_{l+1}^2) & (u_lv_l+u_{l+1}v_{l+1})\end{pmatrix}\cdot\chi_D. \quad (2.55)$$

The integration of this equation is completely analogous to the integration of (2.12).

3. CONVERGENCE OF SERIES FOR PHASE SHIFT

Let us discuss the series for the phase shift, (2.49). Each term $\eta^{(n)}$ in (2.49) can be expanded into its Born series; the $\eta^{(n)}$ starts with the (2^n-1)th order. That is to say, the straightforward Born series is partially summed to yield each $\eta^{(n)}$, and then they are summed in (2.49). The advantage of the new series lies in, that the sum of the first n terms includes all the $(2^{n+1}-2)$ terms in the Born series. Therefore, if $\eta^{(n)}$ can be computed, the series (2.49) converges faster than the straightforward Born series. In fact, the $\eta^{(n)}$ is given by a set of relatively simple recursion formulas, (2.30)–(2.33). At high energies it converges quickly for a nonsingular potential, as will be shown in the next section.

Now, we shall show that the Born series has a finite radius of convergence provided the potential has no singularity stronger than $1/r$ at the origin and is decreasing faster than $1/r^2$ for large r. The convergence is absolute for all energies. A bound to the $\eta^{(n)}$ can be set by using the following bounds to the Riccatti Bessel functions[18]:

$$|u_l(pr)|\leq C\left(\frac{pr}{1+pr}\right)^{l+1},$$
$$|v_l(pr)|\leq C\left(\frac{pr}{1+pr}\right)^{-l}, \quad (3.1)$$

where C is some constant. An auxiliary function $a(r)$ defined by

$$a(r)=C^2\int_0^r \frac{pr'}{1+pr'}|U(r')|dr' \quad (3.2)$$

is used below. The parameter which measures the strength of the potential is given by

$$\alpha=a(\infty)/p. \quad (3.3)$$

The function $\lambda^{(0)}(r)$ is used in the definition of $\eta^{(1)}$; its bound is given by

$$|\lambda^{(0)}(r)|\leq\frac{1}{p}\int_r^{\infty}|u_l||v_l||U(s)|ds\leq\frac{1}{p}\int_0^{\infty}|u_l||v_l||U(s)|ds$$
$$\leq\frac{C^2}{p}\int_0^{\infty}\frac{ps}{1+ps}|U(s)|ds=\alpha, \quad (3.4)$$

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[18] R. G. Newton, J. Math. Phys. **1**, 319 (1960).
[19] For instance, see M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, Inc., New York, 1961), p. 159.

which follows from (2.17), (2.13), and (3.1)–(3.3). Using (3.4), the following bound can be set to $\eta^{(1)}$; from its definition (2.44) we see

$$|\eta^{(1)}|\leq\int_0^\infty \frac{1}{p}u_l^2(pr)|U(r)|\exp[2|\lambda^{(0)}(r)|]dr$$
$$\leq C^2\int_0^\infty \frac{1}{p}\left(\frac{pr}{1+pr}\right)^{2l+2}|U(r)|\exp(2\alpha)dr.$$

Since we have

$$\frac{pr}{1+pr}\leq 1, \tag{3.5}$$

for all r, we may set

$$|\eta^{(1)}|\leq\exp(2\alpha)\cdot C^2\int_0^\infty \frac{r}{1+pr}|U(r)|dr=\alpha\exp(2\alpha). \tag{3.6}$$

Similarly a bound to the function $\xi^{(1)}(r)$, (2.23), can be set by

$$|\xi^{(1)}(r)|\leq\int_0^r \frac{1}{p}u_l^2(pr')|U(r')|\exp[2|\lambda^{(0)}(r')|]dr'$$
$$\leq\exp(2\alpha)\frac{C^2}{p}\int_0^r\left(\frac{pr'}{1+pr'}\right)^{2l+2}|U(r')|dr'$$
$$\leq\exp(2\alpha)\frac{a(r)}{p}\left(\frac{pr}{1+pr}\right)^{2l+1}, \tag{3.7}$$

where the inequality

$$\frac{pr}{1+pr}>\frac{pr'}{1+pr'}\quad\text{for}\quad r>r' \tag{3.8}$$

has been used in deriving the last statement.

In order to set a bound to $\eta^{(2)}$, we have to evaluate a bound to $\lambda^{(1)}(r)$, which is defined by (2.27). Here we can use the following bound to the function $U^{(1)}(r)$, (2.19):

$$|U^{(1)}(r)|\leq|L^{(1)}|+|K^{(1)}|$$
$$<\exp[2|\lambda^{(0)}(r)|]\frac{1}{p}[u_l^2(r)+v_l^2(r)]|U(r)|$$
$$<\frac{2}{p}\exp(2\alpha)|U(r)|C^2\left(\frac{pr}{1+pr}\right)^{-2l}, \tag{3.9}$$

where (3.1), (3.4), and (3.5) have been used. Bounds to $\cos\xi^{(1)}(r)$ and $\sin\xi^{(1)}(r)$ may be set by

$$\begin{aligned}&|\cos\xi^{(1)}(r)|\leq 1,\\&|\sin\xi^{(1)}(r)|\leq|\xi^{(1)}(r)|.\end{aligned} \tag{3.10}$$

Using (2.27), (2.25), (3.9), and (3.10), we have

$$|\lambda^{(1)}(r)|\leq\int_r^\infty|M_{uv}^{(1)}(r')|dr'\leq\int_0^\infty|M_{uv}^{(1)}(r')|dr'$$
$$\leq\int_0^\infty\frac{2}{p}\exp(2\alpha)C^2|U(r)|\left(\frac{pr}{1+pr}\right)^{-2l}|\xi^{(1)}(r)|dr.$$

Taking account of (3.7), we see

$$|\lambda^{(1)}(r)|<\exp(4\alpha)\alpha^2. \tag{3.11}$$

Here use has been made of

$$\frac{1}{n}[a(r)]^n=C^2\int_0^r\frac{pr'}{1+pr'}|U(r')|[a(r')]^{n-1}dr'. \tag{3.12}$$

Using (3.11), a bound to $\eta^{(2)}$, (2.46), is set by

$$|\eta^{(2)}|\leq\int_0^\infty|\xi^{(1)}(r)|^2|U^{(1)}(r)|\exp[2|\lambda^{(1)}(r)|]dr$$
$$\leq\exp\{2[\alpha\exp(2\alpha)]^2\}\int_0^\infty|\xi^{(1)}(r)|^2|U^{(1)}(r)|dr.$$

Taking (3.7) and (3.9) into account, one finds

$$|\eta^{(2)}|\leq\exp[6\alpha+2(\alpha\exp2\alpha)^2]$$
$$\times\int_0^\infty\left(\frac{a(r)}{p}\right)^2\frac{2}{p}C^2|V(r)|\left(\frac{pr}{1+pr}\right)^{2l+2}$$
$$\leq\tfrac{2}{3}\alpha^3\exp[6\alpha+2(\alpha\exp2\alpha)^2]. \tag{3.13}$$

In the derivation of the last statement we have used (3.5) and (3.12) with $n=3$. Similarly a bound to $\xi^{(2)}(r)$ is given by

$$|\xi^{(2)}(r)|\leq\exp[6\alpha+2(\alpha\exp2\alpha)^2]\left(\frac{pr}{1+pr}\right)^{2l+1}$$
$$\times\tfrac{2}{3}[a(r)/p]^3, \tag{3.14}$$

which can be used to evaluate bounds to $\lambda^{(2)}$ and $\eta^{(3)}$.

The computation of bounds to terms of higher order, $\eta^{(n)}$ or $\lambda^{(n)}$, can be made similarly. Let us denote them by $\beta^{(n)}$ and $\gamma^{(n)}$

$$\begin{aligned}&|\eta^{(n)}|<\beta^{(n)},\\&|\lambda^{(n)}(r)|<\gamma^{(n)}.\end{aligned} \tag{3.15}$$

A general form for $\gamma^{(n)}$ may be represented by the following recursion formula

$$\gamma^{(n)}=[\gamma^{(n-1)}]^2(2^{n-2}\exp4\gamma^{(n-1)})/(2^n-1), \tag{3.16}$$

valid for $n>1$. For $n=0$, we should put $\gamma^{(0)}=\alpha$, according to (3.4), and for $n=1$, we put $\gamma^{(1)}=\alpha^2\exp(4\alpha)$ according to (3.11). Using (3.16) for $\gamma^{(n)}$, we have a recursion formula for $\beta^{(n)}$

$$\beta^{(n)}=\beta^{(1)}(\beta^{(n-1)})^2\frac{2}{2^n-1}\exp[2\sum_{m=1}^{n-1}\gamma^{(m)}],\quad n>1. \tag{3.17}$$

For $n=1$, we put

$$\beta^{(1)}=\alpha\exp(2\alpha)$$

according to (3.6). A sufficient condition for the absolute convergence of the series (2.49) is the convergence of the series

$$\sum_{n=1}^{\infty}\beta^{(n)},$$

which we are going to prove now. We first assume that the parameter α is such that the inequality

$$\frac{2^{n-2}}{2^n-1}\exp(4\gamma^{(n-1)})\leq 1 \tag{3.18}$$

holds for $n>1$. From here it follows for $n=2$ that $\gamma^{(1)}<1$. Hence, in conjunction with the recursion formula, (3.16) it is easily established that $\gamma^{(n)}$ is decreasing with increasing n, i.e.,

$$\gamma^{(n)}<\gamma^{(n-1)}<1 \quad \text{for} \quad n>1. \tag{3.19}$$

Therefore, also the left-hand side of (3.18) decreases with increasing n. It follows that (3.18) may be replaced by the simpler condition

$$\tfrac{1}{3}\exp(4\gamma^{(1)})\leq 1, \tag{3.20}$$

which because of $\gamma^{(1)}=\alpha^2\exp(4\alpha)$ becomes

$$\tfrac{1}{3}\exp[4\alpha^2\exp(4\alpha)]\leq 1. \tag{3.21}$$

Call α_0 the value of α for which the equality sign holds, then one finds

$$\alpha_0=0.292. \tag{3.22}$$

Therefore, (3.18) holds for every $\alpha\leq\alpha_0$.

Let us now estimate $\beta^{(n)}$. From (3.17) we have

$$\beta^{(2)}=2(\beta^{(1)})^3[\tfrac{1}{3}(4\gamma^{(1)})]\exp(-2\gamma^{(1)})$$

$$\beta^{(3)}=4(\beta^{(1)})^7[\tfrac{1}{3}\exp(4\gamma^{(1)})]^2[(2/7)\exp(4\gamma^{(2)})] \times\exp[-2(\gamma^{(1)}+\gamma^{(2)})], \text{ etc.}$$

Thus, by use of (3.18), we conclude

$$\beta^{(n)}\leq 2^{n-1}(\beta^{(1)})^{(2^n-1)}. \tag{3.23}$$

Now $\beta^{(1)}<1$, because $\beta^{(1)}=\alpha\exp(2\alpha)=(\gamma^{(1)})^{1/2}<1$ according to (3.6), (3.16), and (3.18). Consequently, we have

$$\sum_1^{\infty}|\eta^{(n)}|<\sum_1^{\infty}\beta^{(n)}\leq\sum_1^{\infty}2^{n-1}(\beta^{(1)})^{(2^n-1)}<\sum_1^{\infty}(\beta^{(1)})^n =\frac{\beta^{(1)}}{1-\beta^{(1)}} \tag{3.24}$$

which establishes the absolute convergence of our series for the phase shift, provided

$$\alpha=C^2\int_0^{\infty}\frac{r|U(r)|}{1+pr}dr\leq\alpha_0 \tag{3.25}$$

is satisfied. It should be emphasized that (3.25) is a sufficient condition for the convergence of the phase-shift series, but by no means a necessary one as will become apparent in the next section. It also follows from (3.24) that if condition (3.25) is fulfilled, the absolute value of the phase shift cannot exceed the value

$$\frac{\beta_0^{(1)}}{1-\beta_0^{(1)}}=\frac{\alpha_0e^{2\alpha_0}}{1-\alpha_0e^{2\alpha_0}}\approx 63°. \tag{3.26}$$

In the proof above some of the inequalities have been adopted in order to simplify the calculations as much as possible. Consequently, we have obtained very stringent conditions, (3.25) and (3.22), for the potential. One would expect that by improving the proof (3.22) could be replaced by a larger number. In fact, the oscillations of the Riccati-Bessel functions for nonvanishing energy have not been taken into account in the foregoing proof. It will be shown in the next section that the condition (3.25) is too restrictive at high energies.

4. HIGH-ENERGY LIMIT OF PHASE SHIFT

The series (2.49) and the discussions in the preceding two sections have been developed for the Schrödinger phase shift. But the simple substitution[20]

$$U(r)\rightarrow\bar{V}(r)=2E_pV(r)-V(r)^2 \tag{4.1}$$

makes them applicable to the Klein-Gordon phase shift; here E_p is the energy for momentum p. The series (2.49) is useful to set limits in various situations, because each term $\eta^{(n)}$ is given in fairly closed form, (2.30)–(2.33); we shall discuss the Klein-Gordon phase shift at high energies.

The high-energy behavior of the phase shift depends critically on whether the potential has any singularity. We shall limit ourselves to the case where the potential is nonsingular and infinitely differentiable for every real r, $0\leq r<\infty$; it is not difficult to include potentials which are piecewise differentiable.

First, we note that the nonvanishing derivatives of the potential at the origin are essential in the derivation of the asymptotic expansion of the amplitude. The asymptotic expansion for large momentum transfer q of the first-order amplitude is

$$f_1(q)=4\pi\int_0^{\infty}\frac{\sin qr}{q}rV(r)dr =8\pi\left[-\frac{1}{q^4}V_0'+2\frac{1}{q^6}V_0'''-\cdots\right], \tag{4.2}$$

where $V_0^{(n)}$ denotes the nth derivative of the potential at the origin. Equation (4.2) is obtained by integration by parts. If all the derivatives of odd order vanish, an asymptotic expansion into inverse powers of q cannot exist but the amplitude may decrease exponentially with increasing q. Such a potential may be called an even potential. It can be expanded in a power series in r^2 around the origin. The distinction between even and

[20] It is to be noted that here the potential is treated as the fourth component of a vector. A change is necessary if an interaction of some other type is to be treated.

noneven potentials has been pointed out by Bethe and Kinoshita in their analysis of Regge poles[9] (for the Schrödinger equation). Also, in reference 7 it has been shown that the leading term of the asymptotic amplitude for a noneven potential is proportional to some inverse power of q. This result indicates that the distinction between even and noneven potentials is retained at all higher orders. This is confirmed by the following proof which is valid both for the Schrödinger equation and Klein-Gordon equation, because if the potential $V(r)$ is even, the right side of (4.1) is also even. We start with the formula

$$\delta_l = -\frac{1}{p}\int_0^\infty \mathcal{V}(r)u_l^2(pr)dr, \tag{4.3}$$

which is the same in form as the first-order phase shift, except that the "effective potential" $\mathcal{V}(r)$ has to be used instead of the potential $\bar{V}(r)$. The effective potential $\mathcal{V}(r)$, in which all higher order effects are taken into account, can be expressed in the form

$$\mathcal{V}(r) = \bar{V}(r)\exp[2\lambda^{(0)}(r)]\sum_{n=1}^{\infty} R^{(n)}(r), \tag{4.4}$$

with $R^{(1)}=1$. Let us calculate $R^{(2)}(r)$. According to (2.46) we have

$$\eta^{(2)} = \int_0^\infty U^{(1)}(r)e^{2\lambda^{(1)}(r)}\sin^2\xi^{(1)}(r)dr$$

$$= \int_0^\infty U^{(1)}(r)e^{2\lambda^{(1)}(r)}\frac{\sin^2\xi^{(1)}(r)}{\xi^{(1)}(r)}dr\int_0^r K^{(1)}(r')dr', \tag{4.5}$$

where the last step follows in virtue of (2.23). If now use is made of

$$\int_0^\infty X(r)dr\int_0^r Y(r')dr' = \int_0^\infty Y(r)dr\int_r^\infty X(r')dr',$$

then $\eta^{(2)}$ becomes

$$\eta^{(2)} = \int_0^\infty K^{(1)}(r)dr\int_r^\infty U^{(1)}(r')e^{2\lambda^{(1)}(r')}\frac{\sin^2\xi^{(1)}(r')}{\xi^{(1)}(r')}dr'.$$

Hence, we obtain

$$R^{(2)} = \int_r^\infty U^{(1)}\exp[2\lambda^{(1)}]\frac{\sin^2\xi^{(1)}}{\xi^{(1)}}dr', \tag{4.6}$$

because of the definition of $K^{(1)}$, (2.19), and (2.13). The generalization to higher orders follows easily from the recursion formula for $\xi^{(n)}$, Eq. (2.31). We find $(n>1)$

$$R^{(n)}(r) = \int_r^\infty U_1(r_1)\frac{\sin^2\xi^{(1)}(r_1)}{\xi^{(1)}(r_1)}dr_1\int_{r_1}^\infty \Lambda_1(r_2)U_2(r_2) \times\frac{\sin^2\xi^{(2)}(r_2)}{\xi^{(2)}(r_2)}dr_2\cdots\int_{r_{n-2}}^\infty \Lambda_{n-2}(r_{n-1})U_{n-1}(r_{n-1}) \times\frac{\sin^2\xi^{(n-1)}(r_{n-1})}{\xi^{(n-1)}(r_{n-1})}dr_{n-1}, \tag{4.7}$$

where the quantities $U_n(r)$ and $\Lambda_n(r)$ are defined by (2.32) and (2.33), respectively. It is now an easy task to prove that for even potentials all the function $R^{(n)}$ are even and thus the effective potential $\mathcal{V}(r)$ is even. First, from (2.17) and (2.13) we see

$$\lambda^{(0)}(r) = \lambda^{(0)}(0) - \frac{1}{p}\int_0^r u_l(pr')v_l(pr')\bar{V}(r')dr'. \tag{4.8}$$

The product $u_l\cdot v_l$ is an odd function of its argument for every l and, therefore, for even potentials the integrand is odd. It follows that the integral is even and, hence, $\lambda^{(0)}(r)$ itself is an even function of r. Next it follows from (2.19) and (2.13) that the function $K^{(1)}(r)$ is even; hence, $\xi^{(1)}(r)$ is odd according to (2.23). Next, $U^{(1)}(r)$ according to its definition (2.19) and (2.13) is an even function of r. Finally, $\lambda^{(1)}(r)$ given by (2.27) is even because $M_{uv}^{(1)}(r)$, (2.25), is odd. We, therefore, conclude that in virtue of (4.6) the function $R^{(2)}$ is even. In a similar way it follows from the recursion formulas for $\lambda^{(n)}$ and $\xi^{(n)}$, (2.30) and (2.31), respectively, that all $\lambda^{(n)}$ are even and all $\xi^{(n)}$, are odd; therefore, the function U_n and Λ_n are even, and by (4.7) all $R^{(n)}$ are even. This concludes the proof that to an even potential corresponds an even effective potential.

Let us now turn to the high-energy limit of the Klein-Gordon phase shift. In reference 7 the WKB phase shifts have been used in order to estimate the contributions from the second- and third-order phase shifts. If we start from the series (2.49), we can develop an analysis similar to the one in reference 7 making use of the exact phase shifts. We would first like to estimate the value of $\eta^{(1)}$ at high energy. With this purpose in mind let us remember[21] that

$$u_l^2 = \tfrac{1}{2} - \tfrac{1}{2}(-1)^l\cos 2pr + O(p^{-2}). \tag{4.9}$$

Therefore, $\eta^{(1)}$ becomes

$$\eta^{(1)} = -\frac{1}{p}\int_0^\infty u_l^2(pr)\bar{V}(r)\exp[2\lambda^{(0)}(r)]dr$$

$$= -\frac{1}{2p}\int_0^\infty \bar{V}(r)\exp[2\lambda^{(0)}(r)]dr + \frac{(-1)^l}{2p}\int_0^\infty \cos 2pr \times\bar{V}(r)\exp[2\lambda^{(0)}(r)]dr + O(p^{-2}). \tag{4.10}$$

Next we wish to find an asymptotic expansion for $\lambda^{(0)}$ which according to (2.17) is given by

$$\lambda^{(0)} = \frac{1}{p}\int_r^\infty u_l(pr')v_l(pr')\cdot\bar{V}(r')dr'.$$

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[21] See, for instance, W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949), p. 22. It should be noted that (4.9) as well as (4.11) and (4.19) below are not asymptotic expansions but break up after a finite number of terms.

Now it is well known[21] that

$$u_l \cdot v_l = -\tfrac{1}{2}(-1)^l \sin 2pr + O(p^{-1}). \tag{4.11}$$

Hence, we obtain

$$\lambda^{(0)}(r) = -\frac{(-1)^l}{2p}\int_r^\infty \sin 2pr' \cdot \bar{V}(r')dr' + O(p^{-2}) \tag{4.12}$$

which after integration by parts becomes

$$\lambda^{(0)}(r) = -(-1)^l \frac{\cos 2pr}{(2p)^2}\bar{V}(r) + O(p^{-2}), \tag{4.13}$$

and, therefore,

$$\exp[2\lambda^{(0)}(r)] = 1 - 2(-1)^l \frac{\cos 2pr}{(2p)^2}\bar{V}(r) + O(p^{-2}). \tag{4.14}$$

Substituting (4.14) into (4.10) then yields the following result for the asymptotic expansion of $\eta^{(1)}$

$$\eta^{(1)} = -\frac{1}{2p}\int_0^\infty \bar{V}(r)dr + \frac{(-1)^l}{2p}\int_0^\infty \cos 2pr \cdot \bar{V}(r)dr + \frac{2}{(2p)^3}(-1)^l \int_0^\infty \cos 2pr \cdot \bar{V}^2(r)dr - \frac{2}{(2p)^3}\int_0^\infty \cos^2 2pr \cdot \bar{V}^2(r)dr + \cdots. \tag{4.15}$$

In virtue of (4.1) the term linear in $1/p$ cancels out (we put $E_p/p = 1$); and therefore, we finally obtain

$$\eta^{(1)} = -\int_0^\infty V(r)dr + O(p^{-2}). \tag{4.16}$$

As $\eta^{(1)}$ contains the first- and second-order phase shifts completely, it follows from (4.16) that at high energy the first-order phase shift is of zeroth order in energy and the second-order phase shift is at least of second order in $1/p$. Actually it can be shown that the second-order phase shift is of third order in $1/p$. This is in agreement with the results of reference 7 [see, Eqs. (5) and (12)].

In order to obtain an idea about the behavior of the higher order phase shifts at high energy, let us expand for the s wave the third-order phase shift asymptotically in energy. The contribution from $\eta^{(2)}$ to $\delta_l^{(3)}$, the third-order phase shift, is given by

$$\delta_{l2}^{(3)} = -2\int_0^\infty (\xi^{(1)})^2(v_l^2 - u_l^2)Vdr = -8\int_0^\infty u_l^2 V dr \int_r^\infty (v_l^2 - u_l^2)Vdr' \times \int_0^{r'} u_l^2 V dr''. \tag{4.17}$$

This follows from (2.46) and (2.19), Next the contribution of $\eta^{(1)}$ to $\delta_l^{(3)}$ is equal to

$$\delta_{l1}^{(3)} = -\frac{1}{p}\int_0^\infty u_l^2[4pV\lambda_{V^2}^{(0)} - 2V^2\lambda_V^{(0)} + 4pV\lambda_V^{(0)2}]dr, \tag{4.18}$$

where $\lambda_V^{(0)}$ and $\lambda_{V^2}^{(0)}$ are both given by (2.17) and (2.13) in which $U(r)$ is replaced by $(2pV)$ and $(-V^2)$, respectively. This follows from (4.1). Explicitly, (4.18) is given by

$$\delta_{l1}^{(3)} = \frac{4}{p}\int_0^\infty u_l^2 V dr \int_r^\infty u_l v_l V^2 dr' + \frac{4}{p}\int_0^\infty u_l^2 V^2 dr \int_r^\infty u_l v_l V dr' - 16\int_0^\infty u_l v_l V dr \int_0^r u_l^2 V dr' \int_{r'}^\infty u_l v_l V dr''. \tag{4.19}$$

We have

$$\delta_l^{(3)} = \delta_{l1}^{(3)} + \delta_{l2}^{(3)}. \tag{4.20}$$

It then follows that the third-order phase shift for the s wave is equal to

$$\delta_0^{(3)} = -8\int_0^\infty \sin^2 pr V dr \int_r^\infty \cos 2pr' V dr' \int_0^{r'} \sin 2pr'' \times V dr'' - \frac{2}{p}\int_0^\infty \sin^2 pr V dr \int_r^\infty \sin 2pr' V^2 dr' - \frac{2}{p}\int_0^\infty \sin^2 pr V^2 dr \int_r^\infty \sin 2pr' V dr' - 4\int_0^\infty \sin 2pr V dr \int_0^r \sin^2 pr' \cdot V dr' \times \int_{r'}^\infty \sin 2pr'' V dr''. \tag{4.21}$$

The asymptotic expansion of $\delta_0^{(3)}$ is obtained by repeated integration by parts of (4.21). The result is

$$\delta_0^{(3)} = 0(p^{-4}). \tag{4.22}$$

Similarly, it can be proved that the higher phase shifts, $\delta^{(n)}$, for $n > 3$, are of higher order than $(1/p)^4$. This means that only the leading term of $\eta^{(1)}$ is retained at infinite energy

$$\delta(\infty) = -\int_0^\infty V(r)dr, \tag{4.23}$$

which is the only term that does not vanish for $p \to \infty$. This result, first derived by Parzen for the Dirac phase shift, is valid for any nonsingular potential, even and noneven. Its validity for the Klein-Gordon phase shift follows from the foregoing proof as well as from the relation[12] between the Dirac and Klein-Gordon phase shifts at high energy. It also follows in virtue of (4.22) that $\eta^{(1)}$ and, therefore, also the first-order phase shift, $\delta^{(1)}$, is a good approximation for the phase shift at high energy in case of nonsingular potentials. This is in agreement with the results of reference 7, where the higher order phase shifts were estimated by the WKB approximation. It was also shown there that for nonsingular, noneven potentials the first-order phase shift was sufficient to derive the leading term in the asymptotic expansion of the scattering amplitude. As far as the calculation of the amplitude for nonsingular potentials at high energy is concerned several questions remain unsettled: (i) Is the higher order WKB approximation of the phase shifts justified? (ii) Is $\eta^{(1)}$ also in the case of even potentials a good approximation of the phase shifts? In order to answer the first question satisfactorily a more detailed analysis of the asymptotic behavior of the phase shifts is necessary. This can be made by a more systematic use of integration by parts than made above. As for the second question we recall that the asymptotic amplitude for a noneven potential has been derived essentially from that part of the phase shift which depends on l in the form (see reference 7)

$$[\text{polynomial in } (l+\tfrac{1}{2})]\times\psi(l+1),$$

where ψ is the logarithmic derivative of the gamma function. On the other hand it has been shown in reference 7 that for an even potential the first-order phase shift does not depend on $\psi(l+1)$. From the first part of the last section of the present paper it follows that this is true to all orders. Therefore, to answer the second question one has to derive an amplitude from phase shifts which do not depend on $\psi(l+1)$.

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Opening Angles of Electron-Positron Pairs

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The cross section for production of a high-energy electron-positron pair of opening angle θ and electron energy ϵ_1, $d^2\sigma(\theta,\epsilon_1)/d\theta d\epsilon_1$ is calculated. Comparison with available experimental data shows good agreement with the present theory. The cross section $d^2\sigma(\theta,\epsilon_1)/d\theta d\epsilon_1$ is shown to be closely related to the cross section for an angle θ_1 between the photon and the electron, $d^2\sigma(\theta_1,\epsilon_1)/d\theta_1 d\epsilon_1$. At high photon energies the functional dependence of $d^2\sigma(\theta,\epsilon_1)/d\theta d\epsilon_1$ on the variable $w=(\epsilon_1\epsilon_2/k)\theta$ is very nearly the same as the functional dependence of $d^2\sigma(\theta_1,\epsilon_1)/d\theta d\epsilon_1$ on the variable $u=\epsilon_1\theta_1$. The experimental method of estimating the energy of a photon creating a pair from the opening angle of the pair is discussed. Formulas for the most probable photon energy for a measured opening angle, including the effect of multiple scattering, are given.

1. INTRODUCTION

THE distribution of the opening angle between the electron and positron of pairs produced by gamma rays has been the subject of many experimental investigations.[1–11] In all cases it has been found that the experimental distribution is considerably more narrow than the theoretical distribution of Borsellino[12] to which the experimental results customarily have been compared. The solution to this puzzle is that Borsellino's cross section does not give the distribution of opening angles for a fixed value of the energy partition between the pair particles, but is rather the distribution function of a certain combination of opening angle and energy partition, viz., the invariant pair energy.

We calculate here the high-energy pair-production cross section as a function of opening angle and energy partition. The resulting distribution of the opening angle is found to be in good agreement with the experimental distributions.

The good agreement between theory and experiment gives one renewed confidence in the method in current use of estimating photon energies by measurement of the

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